Lectured by Dr. H. K. Yadav
Deptt. of Mathematics
Marwari College, Darbhanga.
B.Sc.-part-I(H), paper- II. 09-04-2020.
Analytical Geometry of two dimensions

Q.1. Find the co-ordinates of the radical centre of the three circles $x^2+y^2+4x+7=0$, $2x^2+2y^2+3x+5y+9=0$ and $x^2+y^2+y=0$.
Hence, find the equation of a circle cutting these circles orthogonally.

Soln:- The radical axis of the first and the last circles is obtained by subtracting their equations from each other, i.e.
$$x^2+y^2+4x+7-(x^2+y^2+y)=0 \Rightarrow 4x-y+7=0 \quad \text{--- (I)}$$
The radical axis of the last two circles is
$$2x^2+2y^2+3x+5y+9-2(x^2+y^2+y)=0$$
$$\Rightarrow 3x+3y+9=0 \Rightarrow x+y+3=0 \quad \text{--- II}$$
Solving I and II by cross-multiplication, we get
$$4x-y+7=0$$
$$x+y+3=0$$
$$\frac{x}{-3-7}=\frac{y}{7-12}=\frac{1}{4+1} \Rightarrow \frac{x}{-10}=\frac{y}{-5}=\frac{1}{5} \Rightarrow x=-2 \text{ and } y=-1$$
Hence, the radical centre is $(-2,-1)$.
The radical centre $P(-2,-1)$ will be the centre of the required circle cutting the three given circles orthogonally.
From P draw a tangent PT to any of the given circles, say first, Then $PT=\sqrt{(-2)^2+(-1)^2+4(-2)+7}=\sqrt{4+1-8+7}=\sqrt{4}=2$
This is the radius of the required circle.
Hence, the required equation of the circle is
$$(x+2)^2+(y+1)^2=(2)^2 \quad [\text{from the formula } (x-h)^2+(y-k)^2=a^2]$$
$$\Rightarrow x^2+4x+\cancel{4}+y^2+2y+1=\cancel{4}$$
$$\Rightarrow x^2+y^2+4x+2y+1=0$$
which is the required eqn of the circle

B.Sc - part-I(H) paper-II

Analytical Geometry of two dimensions.

Q.2. Find the general ~~solution~~ equation to the system of co-axal circle containing the circle $x^2+y^2=9$ and the limiting point (2,3). Find also the other limiting point.

Soln: The equation of the point-circle at (2,3) is

$$(x-2)^2+(y-3)^2=0$$

$$x^2-4x+4+y^2-6y+9=0$$

$$\Rightarrow x^2+y^2-4x-6y+13=0 \longrightarrow \text{I}$$

The general equation of the system of co-axal circles containing the given circle $x^2+y^2-9=0$ and the point circle (I) is

$$x^2+y^2-4x-6y+13+\lambda(x^2+y^2-9)=0$$

$$\Rightarrow (1+\lambda)(x^2+y^2)-4x-6y+(13-9\lambda)=0$$

$$\Rightarrow x^2+y^2-2\cdot\frac{2}{1+\lambda}x-2\frac{3}{1+\lambda}y+\frac{13-9\lambda}{1+\lambda}=0 \longrightarrow \text{II}$$

Its centre is $\left(\frac{2}{1+\lambda},\frac{3}{1+\lambda}\right)$

and radius is $\sqrt{\frac{4}{(1+\lambda)^2}+\frac{9}{(1+\lambda)^2}-\frac{13-9\lambda}{1+\lambda}}$.

For limiting points of the system, the radius must be zero.

$$\therefore \frac{4}{(1+\lambda)^2}+\frac{9}{(1+\lambda)^2}-\frac{13-9\lambda}{1+\lambda}=0$$

$$\Rightarrow 13-(13-9\lambda)(1+\lambda)=0 \Rightarrow 13-(13+4\lambda-9\lambda^2)=0$$

$$\Rightarrow \lambda(9\lambda-4)=0 \Rightarrow \lambda=0 \text{ or } \lambda=\frac{4}{9}.$$

When $\lambda=0$, from II, the limiting point is (2,3), which is the given limiting point.

When $\lambda=\frac{4}{9}$, from II, the required limiting point is

$$\left(\frac{18}{13},\frac{27}{13}\right)$$

Lecturered By Dr. H. K. Yadav.
Deptt. of Mathematics
Marwari College, Darbhanga
B.Sc-part-II(H) paper-III
Sequence and Series. Date-09-04-2020.

Q.1. State and prove Raabe's test.

Soln: Statement:- Let us suppose that $u_n > 0$ and that $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$

Then the series $\sum u_n$ is convergent if $l > 1$ and divergent if $l < 1$.

Proof: Let us compare the given series $\sum u_n$ with the auxiliary series

$$\sum v_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

Where nth term. $v_n = \frac{1}{n^p}$

We know that the series $\sum v_n$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

Now, $\frac{v_n}{v_{n+1}} = \frac{1}{n^p} \bigg/ \frac{1}{(n+1)^p}$

$$= \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p = \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \cdots$$

$$= 1 + \frac{p}{n} + \text{terms of the order } \frac{1}{n^2}$$

$$= 1 + \frac{p}{n} + o\left(\frac{1}{n^2}\right)$$

Here, there are two cases's arise

Case I. Let us suppose that $\sum v_n$ is convergent and hence $p > 1$

Then by the comparison test, $\sum u_n$ is convergent if,

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

i.e. if $\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + o\left(\frac{1}{n^2}\right)$

Raabe's test B.sc. part-II, paper-II 09-04-2020
आरोप गया

i.e. if $\frac{u_n}{u_{n+1}} - 1 > \frac{p}{n} + O\left(\frac{1}{n^2}\right)$

i.e. if $n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + O\left(\frac{1}{n}\right)$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq p (>1)$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1.$

Case II Let us suppose that ΣV_n is divergent and hence $p<1$. Then Σu_n is divergent.

if $\frac{u_n}{u_{n+1}} < \frac{V_n}{V_{n+1}}$

i.e. if $Lt\, n\left(\frac{u_n}{u_{n+1}} - 1\right) < p$

i.e. if $Lt\, n\left[\frac{u_n}{u_{n+1}} - 1\right] < 1$ proved

Q.2. Test the convergency of the series $1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + \cdots$

Soln: We have, $u_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{1.2.3\cdots n}$ —— I

$u_{n+1} = \frac{a(a+1)(a+2)\cdots(a+n-1)(a+n)}{1.2.3\cdots n(n+1)}$ —— II

Dividing I by II, we get,

$\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{n+1}{n+a}$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

Hence, ratio test fails and we apply the Raabe's Test.

Now, $\frac{u_n}{u_{n+1}} - 1 = \frac{n+1}{n+a} - 1 = \frac{n+1-n-a}{n+a} = \frac{1-a}{n+a}$

$\therefore n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n(1-a)}{n+a}$ $\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = Lt\, \frac{n}{n+a}(1-a) = 1-a$

Therefore, if $1-a>1$ i.e. $a<0$, the series is convergent and if $1-a<1$ i.e. $a>0$, the series is divergent.

But if $1-a=1$ i.e. $a=0$, the series becomes $1+0+0+0+0+\cdots$ which is clearly convergent.

Thus the series is convergent if $a \leq 0$ and is divergent if $a>0$. Which is the required Test.